On Numerical Methods for PDE-Based Device Simulation: An Introduction

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1. Introduction in **discretization and solving procedure** for basic drift-diffusion model

2. **Mathematical/numerical point of view**

3. **Interplay** between
discretization of space (**grid**) and discretization of differential operators (**matrix**)

4. Importance of **M-matrices**
Content

- **Drift-Diffusion Model**
  An introduction and some analytical properties

- **Solving Procedures**
  Nonlinear and linear solvers

- **Discretization of Drift-Diffusion Model**
  Some FE examples, the box method, and Scharfetter-Gummel boxmethod

- **Grids**
  Some remarks
Drift-Diffusion Model
Drift-Diffusion Model or van Roosbroeck’s equations:
- Describe charge transport in semiconductor devices
- Poisson equation, electron and hole continuity equations (in semiconductors)
- \[-\nabla \cdot (\varepsilon \nabla \phi) = q (p - n + C)\]
- \[q \frac{\partial n}{\partial t} - \nabla \cdot j_n = -q R\]
- \[q \frac{\partial p}{\partial t} + \nabla \cdot j_p = -q R\]
- Completed by electron/hole current equations (using Einstein relation \(D = U_T \mu\))
- \[j_n = -q \mu_n n \nabla \phi_n = q \mu_n (U_T \nabla n - n \nabla \phi)\]
- \[j_p = -q \mu_p p \nabla \phi_p = -q \mu_p (U_T \nabla p + p \nabla \phi)\]
- Physics: validity of equations, modeling of mobility \(\mu\) and recombination \(R\)
  \[\mu = \mu(x, \nabla \phi), \quad R = R(x, n, p, \phi)\]

Not topic of this lecture
DD: Boundary/Interface Conditions

- **Domain of equations**: distinguish semiconductors, insulators, and metals
- **Artificial BCs**: artificially introduced borders or the simulation domain
  \[ \nabla \varphi \cdot \nu = j_n \cdot \nu = j_p \cdot \nu = 0 \]

- **Physical BCs**: contact and material interfaces
  - **Ohmic contacts**:
    \[ np = n_i^2 \]  
    \[ p - n + C = 0 \]  
    thermodynamic equilibrium  
    charge neutrality  
    result in Dirichlet BCs: \( \varphi(x) = \varphi_0(x), n(x) = n_0(x), p(x) = p_0(x) \)
  - **Schottky contacts**: ...
  - **Semiconductor-insulator interfaces**:
    \[ \varepsilon_{\text{semi}} \nabla \varphi_{\text{semi}} = \varepsilon_{\text{insu}} \nabla \varphi_{\text{insu}} \]  
    \[ j_n \cdot \nu = j_p \cdot \nu = 0 \]  
    (neglecting tunneling)
  - **Heterointerfaces**: ...

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Example Structure

Schematic MOSFET model with underlying grid.
Drift-Diffusion Model

**Mathematical View:** (only stationary case)

- **Task:** find functions \( \phi, n, p \) satisfying the above equations
- **Simulation domain** \( \Omega \) : introduce boundary conditions
- Substitute current equations \( j_{n,p} \) into DD equations:
  - *nonlinearly coupled system of elliptic PDEs* (of second order)
- **Typical questions:**
  - Existence of solutions ?
  - Uniqueness of solution ?
  - Is problem well posed (i.e. continuous dependence of solution on ‘data’) ?
- **Nonlinearity:**
  - drift term in the equations
  - Mobility and recombination models
DD: Some Analytical Properties

1. **Existence:**
   The existence of solutions for the whole system is proven for situations close to equilibrium (assuming certain physical models for the problem).

2. **Uniqueness:**
   In general, uniqueness can not be expected as the experience shows.

3. **Layer Behavior:**
   Scalar diffusion-convection-reaction equations with dominant convection exhibit layer behavior (see Roos, Stynes, Tobiska).

4. **Maximum Principle for elliptic PDEs:**
   coming soon
DD: Free Energy and Dissipation Rate

Free Energy:

\[
F(\varphi, n, p) = \frac{1}{2} \int_{\Omega} \varepsilon |\nabla (\varphi - \varphi^*)|^2 dx \\
+ k_B T \int_{\Omega} n \left( \ln \left( \frac{n}{n^*} \right) - 1 \right) + n^* + p \left( \ln \left( \frac{p}{p^*} \right) - 1 \right) + p^* dx
\]

Dissipation Rate:

\[
D(\varphi, n, p) = \int_{\Omega} \mu_n n |\nabla \varphi_n|^2 dx + \int_{\Omega} \mu_p p |\nabla \varphi_p|^2 + k_B T \int_{\Omega} R \ln \left( \frac{np}{n^*p^*} \right) dx
\]

\(F\) is **Lyapunov function** for transient problem under equilibrium boundary conditions and we have:

\[
F(0) - F(t) = \int_{0}^{t} D(\tau) \, d\tau
\]
Inverse Monotonicity of Elliptic Operators

Let $L$ be a **linear second order elliptic differential operator in divergence form**

$$L u := -\nabla \cdot [a(x)\nabla u + b(x)u]$$

Then we have (e.g. Gilbarg, Trudinger, Theorem 9.5):

- **Inverse Monotonicity:**
  \[\{Lu \geq 0 \text{ on } \Omega \text{ and } u \geq 0 \text{ on } \partial\Omega\} \implies u \geq 0 \text{ on } \Omega\]

- **Comparison Theorem:**
  \[\{Lu \geq Lv \text{ on } \Omega \text{ and } u \geq v \text{ on } \partial\Omega\} \implies u \geq v \text{ on } \Omega\]

- **Maximum/Minimum Principle:**
  \[\{Lu \geq 0 \text{ on } \Omega\} \implies \min_{x \in \Omega}(u(x)) = \min_{x \in \partial\Omega}(u(x))\]

Similar results are valid even for **quasilinear** operators.
M-Matrices

Definition (M-Matrix): The real-valued $n\times n$-matrix $A$ is M-matrix if
1. $A_{ii} > 0$ for all $i$,
2. $A_{ij} \leq 0$ for all $i \neq j$,
3. $A$ is invertible and $A^{-1}$ is nonnegative (i.e. $(A^{-1})_{ij} \geq 0$ for all $i$ and $j$).

Remarks:

• Handy sufficient criterion:
  If $A$ fulfills the first two conditions and is irreducibly diagonally dominant (i.e. all variables are connected via nonzero offdiagonals, and $|A_{ii}| \geq \sum_{i \neq j} |A_{ij}|$, and there exists one $i_0$ with strict diagonal dominance), then $A$ is M-matrix.

• M-matrices are (positive) stable, i.e. the initial value problem in $\mathbb{R}^n$
  \[ \dot{x} + Ax = 0, \quad x(0) = x_0 \]
  converges for all initial values $x_0$ against 0.
Stable matrices with nonpositive offdiagonal entries are M-matrices (Horn,Johnson).

• M-matrices are a discrete analogon to the inverse monotonicity of elliptic operators.
Numerical Discretization

**Continuous Problem**: formulated in *infinite* dimensional function spaces

**TASK**: make finite dimensional

**Popular methods:**
- Finite differences
- Finite elements
- Box method

**Necessary steps:**
1. Grid/mesh generation
2. Discretization of the differential operators
3. Solution of nonlinear equations
4. Solution of linear equations
Solution Procedures
Nonlinear Problem

The discretization results in the nonlinear problem in $\mathbb{R}^n$

$$F(u) = \begin{pmatrix} F_\phi(u) \\ F_n(u) \\ F_p(u) \end{pmatrix} = 0 , \ u = (\phi, n, p) \in \mathbb{R}^n$$

Nonlinear equations can only be solved iteratively.
Newton Algorithm

The well-known Newton iteration:
Given a starting point $u_0$, iterate

$$F'(u_n) \cdot (u_{n+1} - u_n) = -F(u_n)$$

Remarks:
• **Quadratic convergence**: For sufficiently good starting points (assuming smooth functions $F$ and an isolated root $u^*$), we have

$$F(u_{n+1}) = F(u_n) + F'(u_n) \cdot (u_{n+1} - u_n) + O(|u_{n+1} - u_n|^2)$$

therefore we conclude

$$|F(u_{n+1})| = O(|u_{n+1} - u_n|^2) = O(|F(u_n)|^2)$$
$$|u_{n+1} - u_n| = O(|F(u_n)|) = O(|u_n - u_{n-1}|^2)$$

• **Modifications of pure Newton**: degradation of quadratic convergence, improvement of **domain of attraction**
Alternative Nonlinear Solution Procedures

**Gummel Iteration:**
- **Iteration:**

\[
\begin{align*}
\varphi_k, n_k, p_k & \text{ given:} \\
F_\varphi(\cdot, n_k, p_k) &= 0 \quad \rightarrow \quad \varphi_{k+1} \\
F_n(\varphi_{k+1}, \cdot, p_k) &= 0 \quad \rightarrow \quad n_{k+1} \\
F_p(\varphi_{k+1}, n_{k+1}, \cdot) &= 0 \quad \rightarrow \quad p_{k+1}
\end{align*}
\]

- **Convergence:** might converge in case of weak coupling of equations

**Multigrid Procedures:**
- **Idea:** solve problem on different grids with different resolutions, thereby resolving **low-frequency** components on coarse grids and **high-frequency** components on fine grids
- **Variants:** on **geometric** level (grid) or on the **algebraic** level (matrix)
Solution of Linear Equations

Consider the linear equation \((A \in M^{n \times n}(\mathbb{R}), b \in \mathbb{R}^n)\):

\[ Au = b \]

Remarks:
1. **Sparsity**: matrices from FD/FE/BM discretizations are sparse, i.e. most entries are zero
2. **Nature of Matrix**: different procedures for specific sparse matrix problems (e.g. band-structured, symmetric, diagonally dominant, structurally symmetric, …)

Two Solver Categories:

- **Direct Methods**:
  - based on **Gauss-algorithm**, perform LU factorization
  - Complexity: dense \(O(N^3)\), sparse 2D \(O(N^{3/2})\), sparse 3D \(O(N^2)\)
  - Experimental memory: 2D about 6 times matrix size, 3D about 20 times

- **Iterative Methods**:
  - splitting methods
  - **Krylov subspace methods** (CG, GMRES)
  - algebraic multigrid
Matrix Condition Number

The condition number of a matrix (Golub, van Loan, ‘Matrix Computations’, 1989)

$$\kappa(A) := ||A|| \cdot ||A^{-1}||$$

characterizes the sensitivity of the perturbated equation

$$(A + \varepsilon F) u_\varepsilon = b + \varepsilon f$$

It can be derived

$$\frac{||u_\varepsilon - u_0||}{||u_0||} \leq \kappa(A) \left( \varepsilon \frac{||F||}{||A||} + \varepsilon \frac{||f||}{||b||} \right) + O(\varepsilon^2)$$

We have machine precision $\varepsilon \approx 10^{-16}$

(ANSI/IEEE Standard 754-1985 for ‘double floating point numbers’: 64 bit – 1 sign bit, 11 exponent bits, 52 fraction bits)

Maximal number of valid digits of solution $u \approx 16 - \log_{10}(\kappa(A))$

Device simulation: matrices are stiff, i.e. large condition numbers
GMRES

Generalized Minimal Residual (GMRES) Method:
Let $x_0, \ldots, x_k$ be given, $r_k := b - Ax_k$ the residuals, and $V_{k+1} := x_0 + \langle r_0, \ldots, r_k \rangle$ a $(k + 1)$-dimensional space. Define $x_{k+1}$ by:

$$\| b - Ax_{k+1} \|_2 = \min_{x \in V_{k+1}} (\| b - Ax \|_2)$$

Remarks:
• Detailed algorithm is technical, omitted here.
• Algorithm requires only matrix-vector products $Ax$, but not the matrix itself.
• The sequence $(x_k)_k$ converges in at most $n$ steps.
• Need to store $k$ vectors to compute $x_{k+1}$.
• GMRES may stagnate (well known, but not really understood).
• A popular variant is the GMRES(m), a restarted GMRES method: stop after $m$ iterations and initialize the procedure again.
• If $A$ is positive definite, GMRES(m) converges for any $m \geq 1$.
• General convergence results for GMRES(m) are not available.
Preconditioning

Idea: Instead of solving $Ax = b$ we solve

$$P_L^{-1}Ax = P_L^{-1}b$$

Remarks:

• $P_L$ should be easier to invert than $A$.

• **Convergence:** If $P_L$ is close to $A$, we have $\|1 - P_L^{-1}A\| < 1$, sufficient for convergence of simple methods.

• **Right preconditioning:** solve $AP_R^{-1}y = b$ for $y$, compute $x = P_R^{-1}y$.

• Right vs left preconditioning:
  - Left preconditioning minimizes the preconditioned residual.
  - Right preconditioning minimizes the unpreconditioned residual.
  - For **ill-conditioned systems** this makes a difference.

Some preconditioning strategies:

• **Incomplete LU factorization ILU** (with/without threshold).

• Think about **physically motivated preconditioners**.
Discretization of the Drift-Diffusion Model
BVP: Strong and Weak Formulation

Elliptic boundary value problem (BVP) of the following form:

\[ Lu := -\nabla \cdot (a \nabla u) + bu = f \quad \text{on } \Omega \]
\[ a \frac{\partial u}{\partial n} = g \quad \text{on } \partial \Omega_N \]
\[ u = 0 \quad \text{on } \partial \Omega_D \]

Strong formulation of the problem: Find a function \( u \in H \) with the above properties.

Alternative: Choose a test function \( v \in H_0 = \{ u \in H : u = 0 \text{ on } \partial \Omega_D \} \), multiply the strong problem and integrate by parts.

Weak formulation of the problem:

Find \( u \in H_D = \{ u \in H : u \text{ satisfies Dirichlet BCs on } \partial \Omega_D \} \) such that for all \( v \in H_0 \) we have

\[ B(u, v) := (a \nabla u, \nabla v) + (bu, v) = (f, v) - \int_{\partial \Omega_N} g \, dS(x) \]
1D Laplace Equation: Standard FE

Laplace equation 1D

\[ Lu := -\nabla \cdot (\nabla u) = f \quad \text{on } \Omega \]
\[ u = 0 \quad \text{on } \partial \Omega \]

Weak formulation: Find \( u \in H^1_0 \) (Sobolev space) with

\[ B(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f v \, dx = (f, v) \]

Standard FE on grid \((x_0, \ldots, x_N)\):

Ansatz: \( u(x) = \sum_j u_j \xi_j(x) \), where \( \xi_i \) is hat function at \( x_i \)

Computation per element \( K = [x_i, x_{i+1}], h_i := x_{i+1} - x_i \)

\[ B^K(\xi_i, \xi_i) = \int_K \left( \frac{1}{h_i} \right)^2 \, dx = \frac{1}{h_i} \]
\[ B^K(\xi_i, \xi_{i+1}) = -\frac{1}{h_i} \]

Element matrix: \( A^K = \begin{pmatrix} 1/h_i & -1/h_i \\ -1/h_i & 1/h_i \end{pmatrix} \)

Global matrix: \( A = \text{tridiag} \left( -1/h_{i-1}, 1/h_{i-1} + 1/h_i, -1/h_i \right) \)

We get a M-matrix
2D Laplace Equation: Standard FE

Laplace equation with homogenous Dirichlet BCs in 2D

\[ B(u,v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f v \, dx = (f,v) \]

Remarks:
1. Bilinear form \( B \) can be evaluated on \( U^h \times U^h \), hence \( B^h \) is uniformly elliptic.
2. The right integral can not be computed exactly for general \( f \in L^2(\Omega) \):
   Ansatz \( f = \sum_j f_j \xi_j(x) \) leads to discrete form \( Mf \)
3. Resulting linear system
   \[ A u = M f \]
4. \( A \) is positive definite, hence stable.
5. \( A \) is not necessarily M-matrix, but we have in 2D:
   For triangulations without obtuse angles, then \( A \) is M-matrix.
6. Mesh geometry determines matrix properties.
7. Similar results hold for the Poisson equation
   \[ B(u,v) = \int_{\Omega} a(x) \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f v \, dx = (f,v) \]
Box Method (BM)

Assumption: Divergence form of operator

\[ Lu(x) = -\nabla \cdot F(x, u) = f(x) \]

and partition of \( \Omega \) into boxes \( B_i \).

Gauss theorem per box \( B_i \):

\[ \int_{B_i} Lu(x) \, dx = -\int_{B_i} \nabla \cdot F(x, u(x)) \, dx = -\int_{\partial B_i} F(x) \cdot \nu_i(x) \, dS(x) \]

Remarks:

- Transform divergence form from volume integral into surface integral
- We need approximation for \( F(x) \cdot \nu_i(x) \) on box boundary.
- Form of boxes not yet specified.
- Relation to FE: The test function is the characteristic function of the box, trial functions are not yet specified.
BM: Voronoi Boxes

Voronoi boxes: defined by mid-perpendicular ‘planes’ of all grid edges:

\[ B_i = \{ x \in \Omega : |x - x_i| \leq |x - x_j| \text{ for all } j \neq i \} \]
BM: Delaunay Property

Delaunay Property:
The (inner of the) circumsphere/circle of each grid element does not contain any grid point.

Remarks:
• Delaunay guarantees overlap-free partitioning of \( \Omega \) with Voronoi boxes.

• Obtuse angles (i.e. \( \geq \pi/2 \)):

\[
s^{T_1}_{i,j} < 0 \ , \ s^{T_2}_{i,j} > 0
\]

Delaunay guarantees

\[
s_{i,j} := s^{T_1}_{i,j} + s^{T_2}_{i,j} \geq 0
\]
BM: Poisson Equation

Poisson Equation:

\[ Lu(x) = -\nabla \cdot (a(x) \nabla u) = g(x) \]

Mid-perpendicular box method:

\[- \int_{B_i} \nabla \cdot (a(x) \nabla u) \, dx = - \int_{\partial B_i} a(x) \nabla u \cdot \nu \, dS(x) \approx - \sum_{j(i)} a_{ij} \frac{u_j - u_i}{|x_j - x_i|} s_{ij} \]

\[ \int_{B_i} g(x) \, dx \approx |B_i| g_i \]

with \( a_{ij} = \left( a(x_j) + a(x_i) \right) / 2 \) some average of \( a \) on the edge.

Remarks:

• **M-matrix property** depends on averaging of \( a \).

• **Laplace operator**: std FE and BM coincide in 2D, but differ in 3D (except for equilateral tetrahedra which do not fill the whole space).
1D Drift-Diffusion: Model Problem

Drift-diffusion operator on the interval \([0; 1]\):
\[-[n' - \varphi'n]' = 0\]
\[n(0) = 0, \quad n(1) = 1\]

and assume \(\varphi' = \beta\) to be constant

- **Exact solution:**
  \[n(x) = \frac{\exp(\beta x) - 1}{\exp(\beta) - 1}\]

- Solution is **strictly monotonously increasing** (independent of sign of \(\beta\))
- Well known: large drift causes problems in discretization, leading to **instabilities**
1D Drift-Diffusion: FD Discretization

Equidistant grid: \( h = x_{i+1} - x_i \)

Gradients on intervals left and right: \( s_- := \frac{n_{i} - n_{i-1}}{h} \) and \( s_+ := \frac{n_{i+1} - n_i}{h} \)

Equation:

\[
- \frac{s_+ - s_-}{h} + \beta \frac{s_+ + s_-}{2} = 0
\]

\[
- \frac{n_{i+1} - n_i + n_{i-1}}{h^2} + \beta \frac{s_{i+1} - s_{i-1}}{2h} = 0
\]

Matrix:

\[
A = \frac{1}{2h^2} \text{tridiag}(-2 - h\beta, +4, -2 + h\beta)
\]

• We get \( \frac{s_+}{s_-} = \left(1 + \frac{h\beta}{2}\right)/\left(1 - \frac{h\beta}{2}\right) \) or in words

The solution oscillates if \( h\beta > 2 \) !!!

• The equation poses requirements grid or discretization
• The resulting matrix is not M-matrix
• The characteristic quantity \( P = 2/\beta \) is called mesh Peclet number
• Some words: upwinding method, exponential fitting
1D Scharfetter-Gummel Discretization

**Assumptions:** $[x_0, x_1]$ interval, $J$ constant current density, and $u := \exp(-\phi)$ the Slotboom variable, then

$$J = -\mu n \phi' = \mu \exp(\phi) u'$$

$\mu$ constant, and $\phi$ linear in $x$, and use notation $\Delta x := x_1 - x_0$

**Solve BVP for $u$:**

$$\Delta u = \int \frac{J}{\mu} \exp\left([\phi_1(x - x_0) - \phi_0(x_1 - x)]/\Delta x\right) dx$$

$$= \cdots = \frac{J \Delta x}{\mu \Delta \phi} \left[\exp(-\phi_0) - \exp(-\phi_1)\right]$$

**Express $J$ in terms of densities:** replace $u_i = \exp(-\phi_i)n_i$, then

$$J = \frac{\mu}{\Delta x} \Delta u \Delta \phi \left[\frac{1}{\exp(-\phi_0) - \exp(-\phi_1)}\right] = \frac{\mu}{\Delta x} \left[\frac{\Delta \phi}{\exp(\Delta \phi) - 1} n_1 + \frac{\Delta \phi}{1 - \exp(-\Delta \phi)} n_0\right]$$

$$= \frac{\mu}{\Delta x} \left[b(\Delta \phi) n_1 - b(-\Delta \phi) n_0\right]$$

where we used the **Bernoulli function** $b(x) := x/(\exp(x) - 1)$.
SG Current Density

Scharfetter-Gummel (SG) approximation

\[ J = \frac{\mu}{\Delta x} [b(\Delta \varphi)n_1 - b(-\Delta \varphi)n_0] \]

Remarks:
- SG reduces for \( \Delta \varphi = 0 \) to pure diffusion.
- Resembles an unsymmetrically weighted diffusion expression (artificial diffusion).
- BM with this SG approximation for \( J \) gives M-matrix independent of \( \Delta \varphi \) because

\[ \frac{\partial J}{\partial n_0} < 0, \quad \frac{\partial J}{\partial n_1} > 0 \]
Discretized Equations

Higher dimensions:
- The SG expression is used in the BM, extending to the **SG-BM**.
- The **one-dimensional** character along grid edges remains.

Discretized equations:

\[
(F_\varphi)_i = \left[ \sum_{j(i)} \varepsilon_{ij} \frac{s_{ij}}{d_{ij}} [\varphi_i - \varphi_j] \right] - |B_i|(p_i - n_i + C_i) = 0
\]

\[
(F_n)_i = \left[ \sum_{j(i)} \mu_{n_{ij}} \frac{s_{ij}}{d_{ij}} [b(\varphi_i - \varphi_j)n_i - b(\varphi_j - \varphi_i)n_j] \right] + |B_i|R_i = 0
\]

\[
(F_p)_i = \left[ \sum_{j(i)} \mu_{p_{ij}} \frac{s_{ij}}{d_{ij}} [b(\varphi_j - \varphi_i)p_i - b(\varphi_i - \varphi_j)p_j] \right] + |B_i|R_i = 0
\]
SG-BM: Discussion

• **No closed theory** is known for the SG-BM.

• SG-BM **guarantees stability** on arbitrary boundary Delaunay meshes (extensively used in practice).

• SG-BM as nonconforming Petrov-Galerkin method.

• SG-BM **is locally and globally dissipative**: the dissipation rate per (non-obtuse) simplex is positive (Gajewski-Gartner).

• **Low convergence order** is expected: experiments with grid adaptation show $O(h^{1/2})$.

• The required boundary Delaunay property is **quite restrictive** (compared to simplex meshes).
Grids
3D Example
Quad-Tree vs Normal-Offsetting

Quad-tree and normal-offsetting mesh with current density.
SG-BM and Current Carrying Edges

Observations

• BM current along edge with one element
  \[ I_{ij}^E = s_{ij}^E J_{ij}^E \]

• SG-BM: element edge current densities \( J_{ij}^E \) are not projections of one element vector \( J^E \)

• Large element edge current densities might not be visible on other edges

• Effect on total current: large \( J_{ij}^E \) with small Voronoi surface \( s_{ij}^E \) not visible

Consequences

• Edges should be aligned parallel and orthogonal to the local current density.

• Highly anisotropic grids are desired in such situations (like channel of a MOSFET).
Grid Effect on Terminal Current

Huge current variations for a MOSFET structure during automatic grid adaptation.

Filled symbols indicate currents at same bias of AGM simulation.

AGM: grid adaptation
REF: fixed grid
Concluding Remarks

• We gave an introduction into discretization and solution strategies for the DD model.

• We emphasized the importance of the M-matrix property, which seems to be indispensable.

• We illustrated the relation between mesh and matrix properties.

• Properties of the continuous problem are not automatically inherited by the discrete problem.
Thank you for your attention!