On Numerical Methods for PDE-Based Device Simulation: An Introduction

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December 2017

1. Introduction in
   discretization and solving procedure
   for basic drift-diffusion model
2. **Mathematical/numerical point of view**
3. **Interplay** between
   discretization of space (**grid**)
   and discretization of differential operators (**matrix**)
4. Importance of **M-matrices**

Content

- **Drift-Diffusion Model**
  An introduction and some analytical properties
- **Solving Procedures**
  Nonlinear and linear solvers
- **Discretization of Drift-Diffusion Model**
  Some FE examples, the box method, and Scharfetter-Gummel boxmethod
- **Grids**
  Some remarks

Drift-Diffusion Model

Drift-Diffusion Model

Drift-Diffusion Model or van Roosbroeck's equations:

- Describe charge transport in semiconductor devices
- **Poisson equation**, electron and hole **continuity equations** (in semiconductors)
-

$$-\nabla \cdot (\varepsilon \nabla \varphi) = q\,(p - n + C)$$
$$\mathrm{q}\frac{\partial n}{\partial t} - \nabla \cdot j_n = -q\,R$$
$$q\frac{\partial p}{\partial t} + \nabla \cdot j_p = -q\,R$$

- Completed by electron/hole **current equations** (using Einstein relation $D = U_T\mu$)

$$j_n = -q\mu_n n\,\nabla\varphi_n = q\,\mu_n(U_T\nabla n - n\nabla\varphi)$$
$$j_p = -q\mu_p p\,\nabla\varphi_p = -q\,\mu_p(U_T\nabla p + \mathrm{p}\nabla\varphi)$$

- **Physics**: validity of equations, modeling of mobility μ and recombination R

  $$\mu = \mu(x, \nabla\varphi) \quad , \quad R = R(x, n, p, \varphi)$$

  Not topic of this lecture

DD: Boundary/Interface Conditions

- **Domain of equations**: distinguish semiconductors, insulators, and metals
- **Artificial BCs**: artificially introduced borders or the simulation domain

$$\nabla\varphi \cdot \nu = j_n \cdot \nu = j_p \cdot \nu = 0$$

- **Physical BCs:** contact and material interfaces
  - **Ohmic contacts:**

    $np = n_i^2$ thermodynamic equilibrium

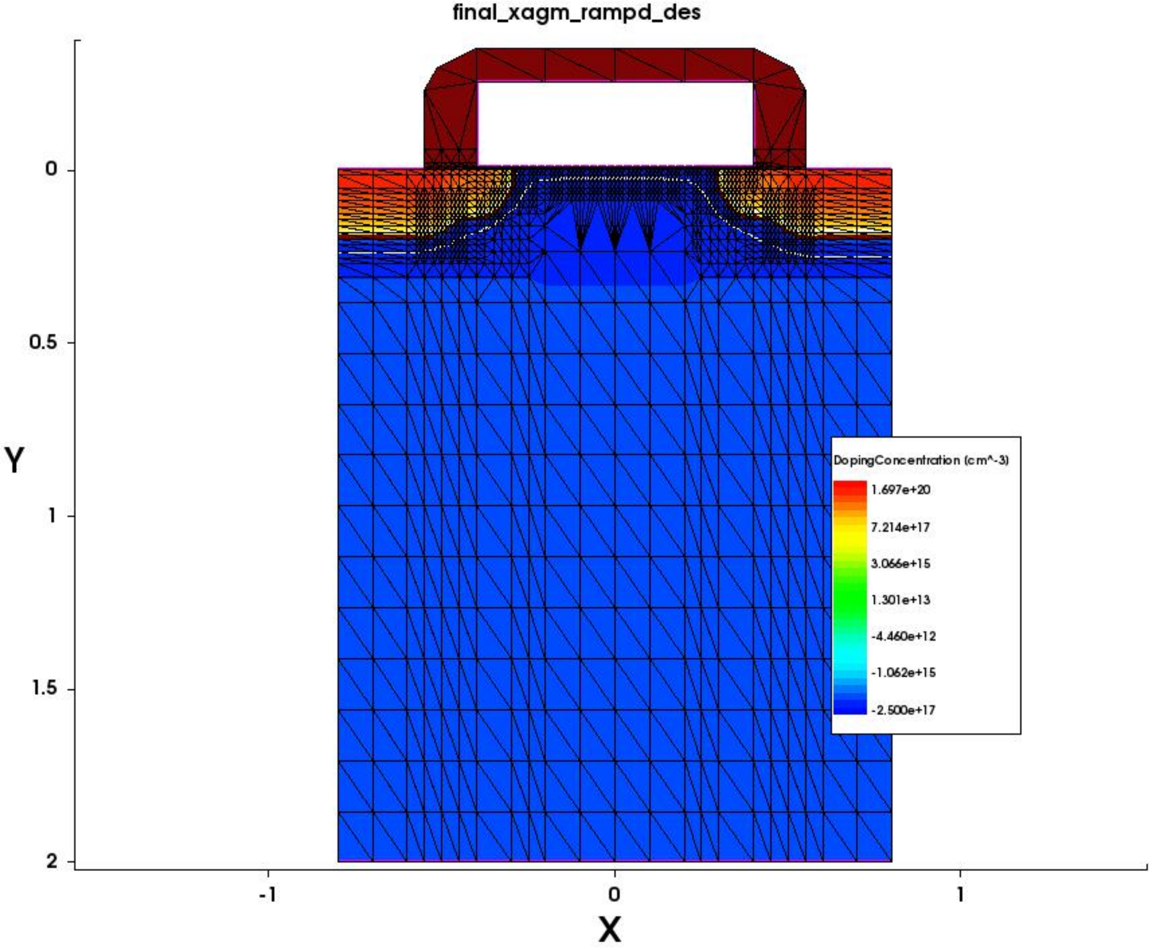
    $p - n + C = 0$ charge neutrality

    result in Dirichlet BCs: $\varphi(x) = \varphi_0(x), n(x) = n_0(x), p(x) = p_0(x)$
  - **Schottky contacts**: …
  - **Semiconductor-insulator interfaces:**

    $\varepsilon_{semi}\nabla\varphi_{semi} = \varepsilon_{insu}\nabla\varphi_{insu}$

    $j_n \cdot \nu = j_p \cdot \nu = 0$ (neglecting tunneling)
  - **Heterointerfaces**: …

Example Structure



Schematic MOSFET model with underlying grid.

Drift-Diffusion Model

Mathematical View: (only stationary case)

- **Task:** find functions φ, n, p satisfying the above equations
- **Simulation domain $\boldsymbol{\Omega}$** : introduce boundary conditions
- Substitute current equations $j_{n,p}$ into DD equations:
  nonlinearly coupled system of elliptic PDEs (of second order)
- **Typical questions:**
  - Existence of solutions ?
  - Uniqueness of solution ?
  - Is problem well posed (i.e. continuous dependence of solution on ‘data’) ?
- **Nonlinearity**:
  - drift term in the equations
  - Mobility and recombination models

DD: Some Analytical Properties

1. **Existence:**
   The existence of solutions for the whole system is proven for situations close to equilibrium (assuming certain physical models for the problem).

2. **Uniqueness:**
   In general, uniqueness can not be expected as the experience shows.

3. **Layer Behavior:**
   Scalar diffusion-convection-reaction equations with dominant convection exhibit layer behavior (see Roos,Stynes,Tobiska).

4. **Maximum Principle for elliptic PDEs:**
   coming soon

DD: Free Energy and Dissipation Rate

Free Energy:

$$F(\varphi, n, p) = \frac{1}{2}\int_\Omega \varepsilon|\nabla(\varphi - \varphi^*)|^2 dx$$
$$+ k_B T \int_\Omega n\left(\ln\left(\frac{n}{n^*}\right) - 1\right) + n^* + p\left(\ln\left(\frac{p}{p^*}\right) - 1\right) + p^* dx$$

Dissipation Rate:

$$D(\varphi, n, p) = \int_\Omega \mu_n n|\nabla\varphi_n|^2 dx + \int_\Omega \mu_p p\left|\nabla\varphi_p\right|^2 + k_B T \int_\Omega R \ln\left(\frac{np}{n^* p^*}\right) dx$$

F is **Lyapunov function** for transient problem under equilibrium boundary conditions and we have:

$$F(0) - F(t) = \int_0^t D(\tau)\, d\tau$$

Inverse Monotonicity of Elliptic Operators

Let L be a **linear second order elliptic differential operator in divergence form**

$$L\,u \;\coloneqq\; -\nabla\cdot[a(x)\nabla u + \boldsymbol{b}(x)u]$$

Then we have (e.g. Gilbarg, Trudinger, Theorem 9.5):

- **Inverse Monotonicity:**

$$\{Lu \geq 0 \text{ on } \Omega \text{ and } u \geq 0 \text{ on } \partial\Omega\} \quad \Longrightarrow \quad u \geq 0 \text{ on } \Omega$$

- **Comparison Theorem:**

$$\{Lu \geq Lv \text{ on } \Omega \text{ and } u \geq v \text{ on } \partial\Omega\} \quad \Longrightarrow \quad u \geq v \text{ on } \Omega$$

- **Maximum/Minimum Principle:**

$$\{Lu \geq 0 \text{ on } \Omega\ \} \quad \Longrightarrow min_{x\in\Omega}\big(u(x)\big) = min_{x\in\partial\Omega}\big(u(x)\big)$$

Similar results are valid even for **quasilinear** operators.

M-Matrices

Definition (M-Matrix): The real-valued nxn-matrix A is **M-matrix** if

1. $A_{ii} > 0$ for all i,
2. $A_{ij} \leq 0$ for all $i \neq j$,
3. A is invertible and A^{-1} is nonnegative (i.e. $(A^{-1})_{ij} \geq 0$ for all i and j).

Remarks:

- **Handy sufficient criterion:**
  If A fulfills the first two conditions and is irreducibly diagonally dominant (i.e. all variables are connected via nonzero offdiagonals, and $|A_{ii}| \geq \sum_{i \neq j} |A_{ij}|$, and there exists one $i_0$with strict diagonal dominance), then A is M-matrix.
- M-matrices are (positive) **stable**, i.e. the initial value problem in $\mathbb{R}^n$

$$\dot{x} + Ax = 0 \qquad , \qquad x(0) = x_0$$

  converges for all initial values x_0 against 0.
  Stable matrices with nonpositive offdiagonal entries are M-matrices (Horn,Johnson).
- M-matrices are a **discrete analogon to the inverse monotonicity** of elliptic operators.

Numerical Discretization

Continuous Problem: formulated in **infinite** dimensional function spaces

TASK: make finite dimensional

Popular methods:

- Finite differences
- Finite elements
- Box method

Necessary steps:

1. Grid/mesh generation
2. Discretization of the differential operators
3. Solution of nonlinear equations
4. Solution of linear equations

Solution Procedures

Nonlinear Problem

The discretization results in the nonlinear problem in $\mathbb{R}^n$

$$F(u) = \begin{pmatrix} F_\varphi(u) \\ F_n(u) \\ F_p(u) \end{pmatrix} = 0 \quad , u = (\varphi, n, p\ ) \in \mathbb{R}^n$$

Nonlinear equations can only be solved iteratively.

Newton Algorithm

The well-known Newton iteration:

Given a starting point u_0, iterate

$$F'(u_n) \cdot (u_{n+1} - u_n) = -F(u_n)$$

Remarks:

- **Quadratic convergence**: For sufficiently good starting points (assuming smooth functions F and an isolated root u^*), we have

  $$F(u_{n+1}) = F(u_n) + F'(u_n) \cdot (u_{n+1} - u_n) + O(|u_{n+1} - u_n|^2)$$

  therefore we conclude

  $$|F(u_{n+1})| = O(|u_{n+1} - u_n|^2) = O(|F(u_n)|^2)$$
  $$|u_{n+1} - u_n| = O(|F(u_n)|) = O(|u_n - u_{n-1}|^2)$$

- **Modifications of pure Newton:**
  degradation of quadratic convergence, improvement of **domain of attraction**

Alternative Nonlinear Solution Procedures

Gummel Iteration:

- **Iteration:**

φ_k, n_k, p_k given:

$$F_\varphi(\ \cdot, n_k, p_k) = 0 \quad \longrightarrow \quad \varphi_{k+1}$$

$$F_n(\varphi_{k+1}, \ \cdot, p_k) = 0 \quad \longrightarrow \quad n_{k+1}$$

$$F_p(\varphi_{k+1}, n_{k+1}, \ \cdot\ ) = 0 \quad \longrightarrow \quad p_{k+1}$$

- **Convergence:** might converge in case of weak coupling of equations

Multigrid Procedures:

- **Idea:** solve problem on different grids with different resolutions, thereby resolving **low-frequency** components on coarse grids and **high-frequency** components on fine grids
- **Variants**: on **geometric** level (grid) or on the **algebraic** level (matrix)

Solution of Linear Equations

Consider the linear equation $(A \in M^{nxn}(\mathbb{R}), b \in \mathbb{R}^n)$:

$$Au = b$$

Remarks:

1. **Sparsity:** matrices from FD/FE/BM discretizations are sparse, i.e. most entries are zero
2. **Nature of Matrix**: different procedures for specific sparse matrix problems (e.g. band-structured, symmetric, diagonally dominant, structurally symmetric, …)

Two Solver Categories:

- Direct Methods:
  - based on **Gauss-algorithm**, perform LU factorization
  - Complexity: dense $O(N^3)$, sparse 2D $O(N^{3/2})$, sparse 3D $O(N^2)$
  - Experimental memory: 2D about 6 times matrix size, 3D about 20 times

- Iterative Methods:
  - splitting methods
  - **Krylov subspace methods** (CG, GMRES)
  - algebraic multigrid

Matrix Condition Number

The **condition number** of a matrix (*Golub, van Loan, 'Matrix Computations', 1989*)

$$\kappa(A) \;\coloneqq\; ||A|| \cdot ||A^{-1}||$$

characterizes the sensitivity of the perturbated equation

$$(A + \varepsilon F)\, u_\varepsilon = \; b + \varepsilon f$$

It can be derived

$$\frac{||u_\varepsilon \; - u_0\,||}{||u_0||} \leq \kappa(A)\left(\varepsilon\frac{||F||}{||A||} + \varepsilon\frac{||f||}{||b||}\right) + \; O(\varepsilon^2)$$

We have machine precision $\varepsilon \;\approx 10^{-16}$

(ANSI/IEEE Standard 754-1985 for 'double floating point numbers': 64 bit – 1 sign bit, 11 exponent bits, 52 fraction bits)

Maximal number of valid digits of solution u $\quad\approx\quad 16\; - \log_{10}(\kappa(A))$

Device simulation: matrices are **stiff**, i.e. large condition numbers

GMRES

Generalized Minimal Residual (GMRES) Method:

Let $x_0, \dots, x_k$ be given, $r_k := b - Ax_k$ the residuals, and $V_{k+1} := x_0 + \langle\{r_0, \dots, r_k\}\rangle$ a $(k+1)$-dimensional space. Define x_{k+1} by:

$$\| b - Ax_{k+1} \|_2 = min_{x \in V_{k+1}}(\| b - Ax \|_2)$$

Remarks:

- Detailed algorithm is **technical**, omitted here.
- Algorithm requires **only matrix-vector products** Ax , but not the matrix itself.
- The sequence $(x_k)_k$ converges in at most n steps.
- Need to store k vectors to compute x_{k+1}.
- GMRES may **stagnate** (well known, but not really understood).
- A popular variant is the **GMRES(m)**, a **restarted GMRES** method: stop after m iterations and initialize the procedure again.
- If A is positive definite, GMRES(m) converges for any $m \geq 1$.
- General convergence results for GMRES(m) are **not available**.

Preconditioning

Idea: Instead of solving $Ax = b$ we solve

$$P_L^{-1} A\, x = P_L^{-1} b$$

Remarks:

- P_L should be easier to invert than A.
- **Convergence**: If P_L is close to A, we have $\| 1 - P_L^{-1} A \| < 1$, sufficient for convergence of simple methods.
- **Right preconditioning**: solve $A P_R^{-1} y = b$ for y, compute $x = P_R^{-1} y$.
- Right vs left preconditioning:
  Left preconditioning minimizes the preconditioned residual.
  Right preconditioning minimizes the unpreconditioned residual.
  For **ill-conditioned systems** this makes a difference.

Some preconditioning strategies:

- **Incomplete LU factorization ILU** (with/without threshold).
- Think about **physically motivated preconditioners**.

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Discretization of the Drift-Diffusion Model

BVP: Strong and Weak Formulation

Elliptic boundary value problem (BVP) of the following form:

$$Lu := -\nabla \cdot (a\nabla u) + bu = f \qquad \text{on } \Omega$$

$$a\frac{\partial u}{\partial n} = g \qquad \text{on } \partial\Omega_N$$

$$u = 0 \qquad \text{on } \partial\Omega_{\mathrm{D}}$$

Strong formulation of the problem: Find a function $u \in H$ with the above properties.

Alternative: Choose a test function $v \in H_0 = \{u \in H : u = 0 \; on \; \partial\Omega_D\}$, multiply the strong problem and integrate by parts.

Weak formulation of the problem:

Find $u \in H_D = \{u \in H : u \; satisfies \; Dirichlet \; BCs \; on \; \partial\Omega_D\}$ such that for all $v \in H_0$ we have

$$B(u, v) := (a\nabla u, \nabla v) + (bu, v) = (f, v) - \int_{\partial\Omega_N} g \; dS(x)$$

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1D Laplace Equation: Standard FE

Laplace equation 1D

$$Lu := -\nabla \cdot (\nabla u) = f \qquad \text{on } \Omega$$

$$u = 0 \qquad \text{on } \partial\Omega$$

Weak formulation: Find $u \in H_0^1$ (Sobolev space) with

$$B(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\ dx = \int_\Omega f v\ dx = (f, v)$$

Standard FE on grid $(\boldsymbol{x_0, \dots, x_N})$:

Ansatz: $u(x) = \sum_j u_j \xi_j(x)$, where ξ_i is hat function at x_i

Computation per element $K = [x_i, x_{i+1}]$, $h_i := x_{i+1} - x_i$

$$B^K(\xi_i, \xi_i) = \int_K \left(\frac{1}{h_i}\right)^2 dx = \frac{1}{h_i}$$

$$B^K(\xi_i, \xi_{i+1}) = -\frac{1}{h_i}$$

Element matrix: $A^K = \begin{pmatrix} 1/h_i & -1/h_i \\ -1/h_i & 1/h_i \end{pmatrix}$

Global matrix: $A = tridiag\,(-1/h_{i-1}, 1/h_{i-1} + 1/h_i, -1/h_i)$

We get a **M-matrix**

2D Laplace Equation: Standard FE

Laplace equation with homogenous Dirichlet BCs in 2D

$$B(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\ dx = \int_\Omega fv\ dx = (f,v)$$

Remarks:

1. Bilinear form B can be evaluated on $U^h \times U^h$, hence B^h is uniformly elliptic.
2. The right integral can not be computed exactly for general $f \in L^2(\Omega)$:
   Ansatz $f = \sum_j f_j \xi_j(x)$ leads to discrete form Mf
3. **Resulting linear system**

$$A\,u = M\,f$$

4. A is **positive definite**, hence **stable**.
5. A is **not necessarily M-matrix**, but we have in 2D:
   For triangulations without obtuse angles, then A is **M-matrix**.
6. **Mesh geometry determines matrix properties.**
7. Similar results hold for the **Poisson equation**

$$B(u,v) = \int_\Omega a(x)\, \nabla u(x) \cdot \nabla v(x)\ dx = \int_\Omega fv\ dx = (f,v)$$

Box Method (BM)

Assumption: Divergence form of operator

$$Lu\ (x) = -\nabla \cdot \boldsymbol{F}(x,u) = f(x)$$

and partition of Ω into boxes B_i.

Gauss theorem per box B_i:

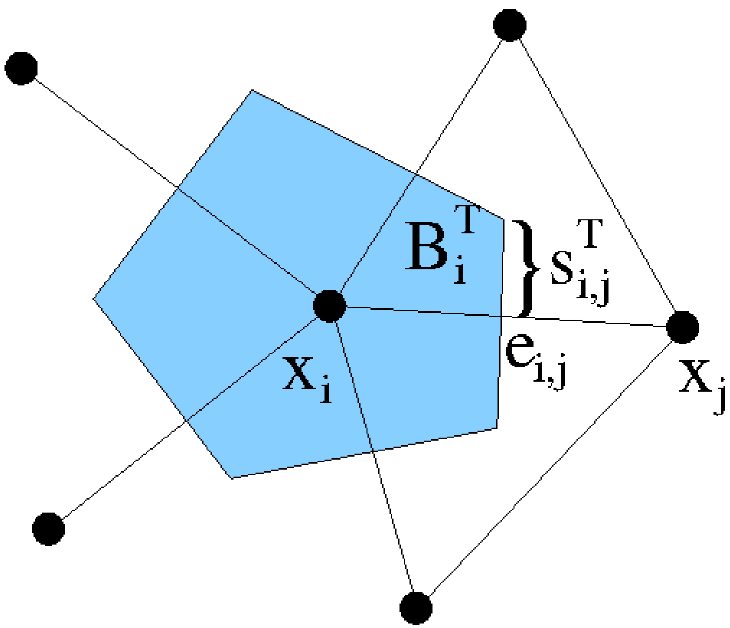
$$\int_{\mathrm{B_i}} Lu(x)\ dx = -\int_{B_i} \nabla \cdot \boldsymbol{F}(x,u(x))\ dx = -\int_{\partial B_i} \boldsymbol{F}(x) \cdot \nu_i(x)\ dS(x)$$

Remarks:

- Transform divergence form from volume integral into surface integral
- We need approximation for $\boldsymbol{F}(x) \cdot \nu_i(x)$ on box boundary.
- Form of boxes not yet specified.
- Relation to FE: The test function is the characteristic function of the box, trial functions are not yet specified.

BM: Voronoi Boxes

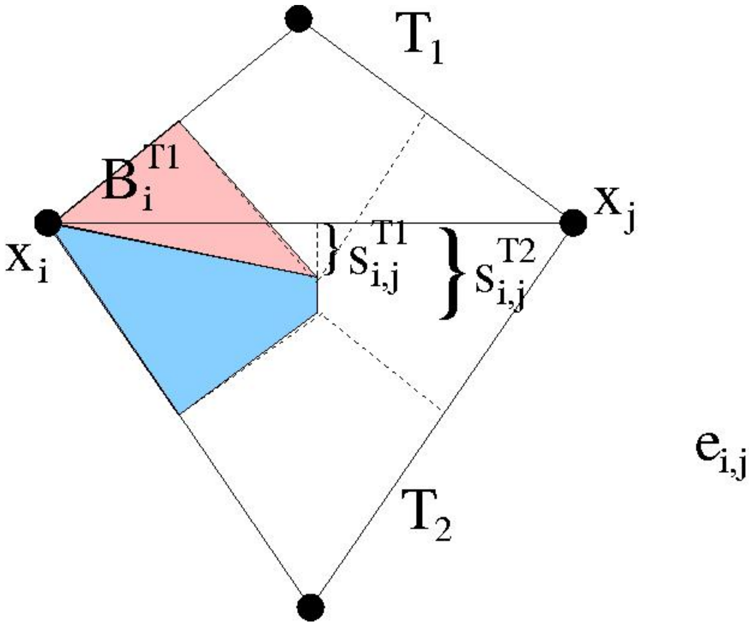
Box method with grid vertices (●)
and dual Voronoi grid (blue)



Voronoi boxes: defined by mid-perpendicular 'planes' of all grid edges:

$$B_i = \{ x \in \Omega \ : \ |x - x_i| \leq |x - x_j| \ for\ all\ j \neq i \}$$

BM: Delaunay Property



Delaunay Property:

The (inner of the) circumsphere/circle of each grid element does not contain any grid point.

Remarks:

- Delaunay guarantees overlap-free partitioning of Ω with Voronoi boxes.
- Obtuse angles (i.e $\geq \pi/2$):

$$s^{T_1}{}_{i,j} < 0 \ , \quad s^{T_2}{}_{i,j} > 0$$

Delaunay guarantees

$$s_{i,j} \ \coloneqq s^{T_1}{}_{i,j} + s^{T_2}{}_{i,j} \ \geq 0$$

BM: Poisson Equation

Poisson Equation:

$$Lu(x) = -\nabla \cdot (a(x)\nabla u\,) = g(x)$$

Mid-perpendicular box method:

$$-\int_{\mathrm{B_i}} \nabla \cdot (a(x)\nabla u\,)dx = -\int_{\partial \mathrm{B_i}} a(x)\nabla \mathrm{u}(\mathrm{x}) \cdot \nu_i \;\, dS(x) \approx -\sum_{j(i)} a_{ij} \frac{u_j - u_i}{\left|x_j - x_i\right|}\, s_{ij}$$

$$\int_{B\_i} g(x)\;\, dx \;\approx |B_i|\; g_i$$

with $a_{ij} = \left(a\left(x_j\right) + a(x_i)\right)/2$ some average of a on the edge.

Remarks:

- **M-matrix property** depends on averaging of a.
- **Laplace operator**: std FE and BM coincide in 2D, but differ in 3D (except for equilateral tetrahedra which do not fill the whole space).

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1D Drift-Diffusion: Model Problem

Drift-diffusion operator on the interval $[0;1]$:

$$-[n' - \varphi' n]' = 0$$

$$n(0) = 0 \ \ , \ n(1) = 1$$

and assume $\varphi' = \beta$ to be constant

o **Exact solution**: $$n(x) = \frac{\exp(\beta x) - 1}{\exp(\beta) - 1}$$

o Solution is **strictly monotonously increasing** (independent of sign of β)

o Well known: large drift causes problems in discretization, leading to **instabilities**

1D Drift-Diffusion: FD Discretization

Equidistant grid: $h = x_{i+1} - x_i$

Gradients on intervals left and right: $s_- := \frac{n_i - n_{i-1}}{h}$ and $s_+ := \frac{n_{i+1} - n_i}{h}$

Equation:

$$-\frac{s_+ - s_-}{h} + \beta \frac{s_+ + s_-}{2} = 0$$

$$-\frac{n_{i+1} - n_i + n_{i-1}}{h^2} + \beta \frac{s_{i+1} - s_{i-1}}{2h} = 0$$

Matrix:

$$A = \frac{1}{2h^2} tridiag(-2 - h\beta, +4, -2 + h\beta)$$

- We get $\frac{s_+}{s_-} = \left(1 + \frac{h\beta}{2}\right) / \left(1 - \frac{h\beta}{2}\right)$ or in words

The solution oscillates if $h\beta > 2$!!!

- The equation poses requirements grid or discretization
- The resulting matrix is **not M-matrix**
- The characteristic quantity $P = 2/\beta$ is called **mesh Peclet number**
- Some words: upwinding method, exponential fitting

1D Scharfetter-Gummel Discretization

Assumptions: $[x_0, x_1]$ interval, J constant current density, and $u \coloneqq \exp(-\phi)$ the Slotboom variable, then

$$J = -\mu n\, \phi' = \mu \exp(\varphi) u'$$

μ constant, and φ linear in x, and use notation $\Delta x \coloneqq x_1 - x_0$

Solve BVP for u:

$$\Delta u = \int \frac{J}{\mu} \exp([-\varphi_1(x - x_0) - \varphi_0(x_1 - x)]/\Delta x)\ dx$$
$$= \cdots = \frac{J}{\mu} \frac{\Delta x}{\Delta \varphi} [\exp(-\varphi_0) - \exp(-\varphi_1)]$$

Express J in terms of densities: replace $u_i = \exp(-\varphi_i) n_i$, then

$$J = \frac{\mu}{\Delta x} \Delta u\, \Delta\varphi \left[\frac{1}{\exp(-\varphi_0) - \exp(-\varphi_1)}\right] = \frac{\mu}{\Delta x}\left[\frac{\Delta\varphi}{\exp(\Delta\varphi) - 1} n_1 + \frac{\Delta\varphi}{1 - \exp(-\Delta\varphi)} n_0\right]$$

$$= \frac{\mu}{\Delta x} [b(\Delta\varphi) n_1 - b(-\Delta\varphi) n_0]$$

where we used the **Bernoulli function** $b(x) \coloneqq x/(\exp(x) - 1)$.

SG Current Density

Scharfetter-Gummel (SG) approximation

$$J = \frac{\mu}{\Delta x}\left[b(\Delta\varphi)n_1 - b(-\Delta\varphi)n_0\right]$$

Remarks:

- SG reduces for $\Delta\varphi = 0$ to pure diffusion.
- Resembles an unsymmetrically weighted diffusion expresion (artificial diffusion).
- BM with this SG approximation for J gives M-matrix independent of $\Delta\varphi$ because

$$\frac{\partial J}{\partial n_0} < 0 \quad , \qquad \frac{\partial J}{\partial n_1} > 0$$

Discretized Equations

Higher dimensions:

- The SG expression is used in the BM, extending to the **SG-BM**.
- The **one-dimensional** character along grid edges remains.

Discretized equations:

$$\left(F_\varphi\right)_i = \left[\sum_{j(i)} \varepsilon_{ij} \frac{s_{ij}}{d_{ij}} \left[\varphi_i - \varphi_j\right]\right] - |\mathrm{B_i}|(p_i - n_i + C_i) = 0$$

$$(F_n)_i = \left[\sum_{j(i)} \mu_{ij}^n \frac{s_{ij}}{d_{ij}} \left[b\left(\varphi_i - \varphi_j\right)n_i - b\left(\varphi_j - \varphi_i\right)n_j\right]\right] + |\mathrm{B_i}|R_i = 0$$

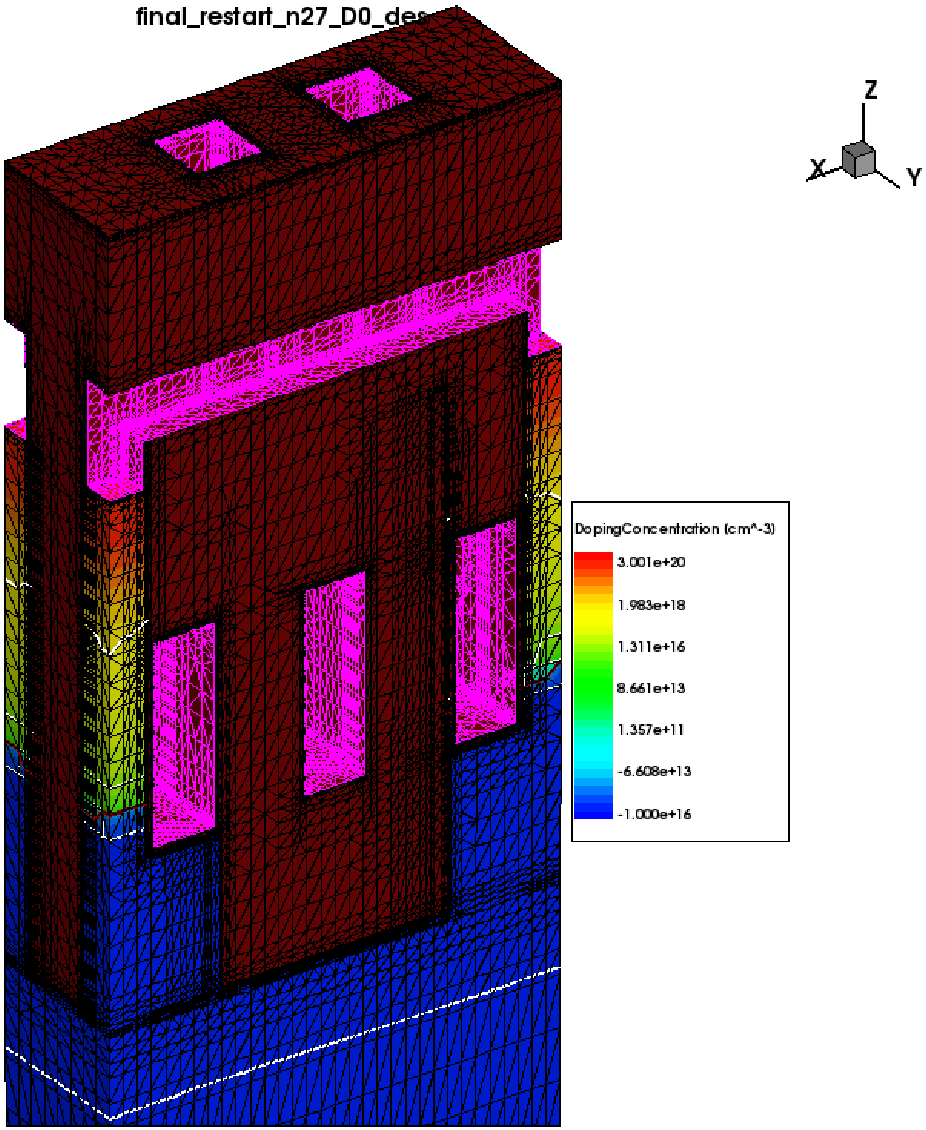
$$\left(F_p\right)_i = \left[\sum_{j(i)} \mu_{ij}^p \frac{s_{ij}}{d_{ij}} \left[b\left(\varphi_j - \varphi_i\right)p_i - b\left(\varphi_i - \varphi_j\right)p_j\right]\right] + |\mathrm{B_i}|R_i = 0$$

SG-BM: Discussion

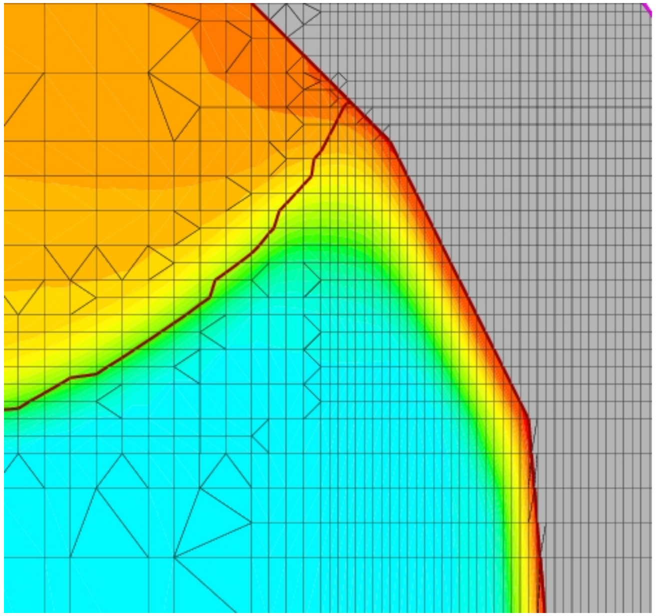
- **No closed theory** is known for the SG-BM.
- SG-BM **guarantees stability** on arbitrary boundary Delaunay meshes (extensively used in practice).
- SG-BM as nonconforming Petrov-Galerkin method.
- SG-BM **is locally and globally dissipative**: the dissipation rate per (non-obtuse) simplex is positive (Gajewski-Gartner).
- **Low convergence order** is expected: experiments with grid adaptation show $O(h^{1/2})$.
- The required boundary Delaunay property is **quite restrictive** (compared to simplex meshes).

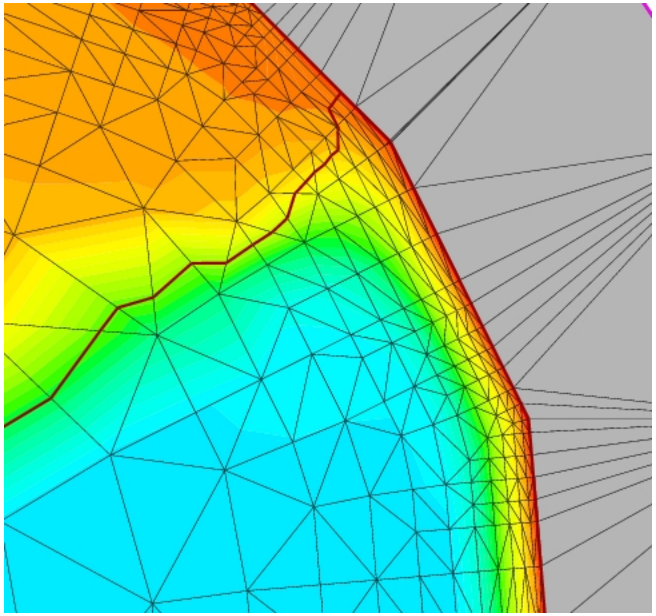
Grids

3D Example


final_restart_n27_D0_des
Z
X
Y
DopingConcentration (cm^-3)
3.001e+20
1.983e+18
1.311e+16
8.661e+13
1.357e+11
-6.608e+13
-1.000e+16

Quad-Tree vs Normal-Offsetting





Quad-tree and normal-offsetting mesh with current density.

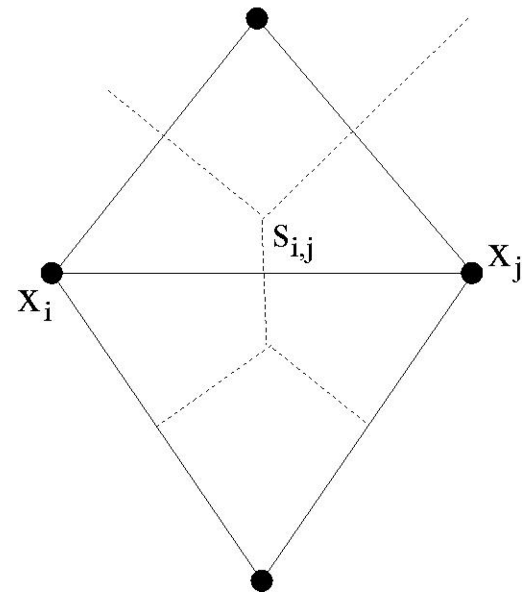
SG-BM and Current Carrying Edges

Observations

- BM current along edge with one element

$$I_{ij}^{E} = s_{ij}^{E}\, J_{ij}^{E}$$

- SG-BM: element edge current densities J_{ij}^{E} are **not projections** of one element vector $\boldsymbol{J}^{E}$
- Large element edge current densities might not be visible on other edges
- Effect on total current: large J_{ij}^{E} with small Voronoi surface s_{ij}^{E} not visible

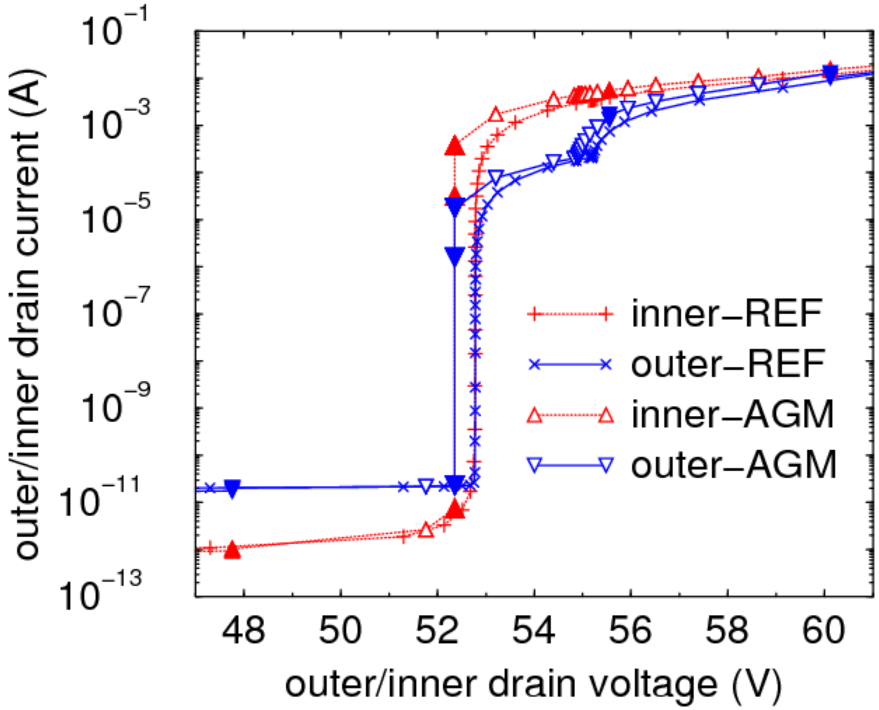


Edge with Voronoi surface

Consequences

- Edges should be **aligned parallel and orthogonal to the local current density.**
- Highly **anisotropic grids** are desired in such situations (like channel of a MOSFET).

Grid Effect on Terminal Current



Huge current variations

for a MOSFET structure
during automatic grid adaptation.

Filled symbols indicate currents at same bias of AGM simulation.

AGM: grid adapation
REF: fixed grid

Concluding Remarks

- We gave an **introduction** into discretization and solution strategies for the DD model.
- We emphasized the importance of the **M-matrix property**, which seems to be indispensable.
- We illustrated the **relation between mesh and matrix** properties.
- Properties of the continuous problem are **not automatically inherited by the discrete problem.**

Thank you for your attention !